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Invariants $\mathcal{L}$ pour $GL_3(\mathbb{Q}_p)$ et cohomologie de l'espace de Drinfel'd de dimension 2

The aim of this talk is to start from certain 3-dimensional $G_{\mathbb{Q}_p}$-representations (semi-stable) and associate to them some complexes of locally analytic representations. We then show how to reconstruct the (φ, N) filtered modules of these representations with these complexes and the de Rham complex on the Drinfel'd space.

$\boxed{K/\mathbb{Q}_p^{<\infty} \text{ coeff field}}$

First, in the $GL_2(\mathbb{Q}_p)$-case, 2-dim semi-stable non crystalline representations are parametrized by $(k, \mathcal{L})$, $k \in \mathbb{Z}$, $\mathcal{L} \in K$ (and a twist...)
$D(k,\mathcal{L}) = D_{st}^{*}(V(k,\mathcal{L}))$.

The p-adic Langlands correspondance associates to $D(k,\mathcal{L})$ an unitary representation $B(k,\mathcal{L})$, completion of $St \otimes \mathrm{Sym}^{k-2} K^2$ for a lattice.

To obtain this lattice, there is a method of Breuil: construct $\Sigma(k,\mathcal{L})$, a locally analytic representation containing $St \otimes \mathrm{Sym}^{k-2} K^2$. $B(k,\mathcal{L})$ is then the universal unitary completion of $\Sigma(k,\mathcal{L})$.

We want construct a $\Sigma(\vec{w}, \vec{\mathcal{L}})$ for semi 3-dim semi-stable representations.

Let $\vec{w} = (0 < w_1 < w_2) \in \mathbb{Z}^3$

$\vec{\mathcal{L}} \in K^3$ $\qquad |\vec{w}| = w_1 + w_2$

$\quad (\mathcal{L}, \mathcal{L}', \mathcal{L}'')$.

The (φ, N)-filtered modules interesting us are of

$$D(\vec{w}, \vec{\mathcal{L}}) = K e_0 \xleftarrow{N} \oplus K e_1 \xleftarrow{N} \oplus K e_2$$

$$\varphi = p^{\frac{|\vec{w}|}{3} - 1} \oplus p^{\frac{|\vec{w}|}{3}} \oplus p^{\frac{|\vec{w}|}{3} + 1}$$

$$\mathrm{Fil}^i = \begin{cases} 0 & i \geq w_2 + 1 \\ K(e_2 + \mathcal{L}' e_1 + \mathcal{L}'' e_0) & w_1 + 1 \leq i \leq w_2 \\ \mathrm{Fil}^{w_1+1} \oplus K(e_1 + \mathcal{L} e_0) & 1 \leq i \leq w_1 \\ D(\vec{w}, \vec{\mathcal{L}}) & i \leq 0 \end{cases}$$

All of these are admissible (φ, N)-filtered modules.

We would like to have a L A representation $\Sigma(\vec{w}, \vec{\mathcal{L}})$ having good properties. In fact, we construct a complex of such locally analytic representations, it is so an objekt living in a derived category.

<u>Remark</u>: There is nothing surprising in that. ~~The~~ Breuil's construction put the $\mathcal{L}$ in a Ext^1. Ours

~~view~~ permits to view $\mathcal{L}''$ in an Ext^2. It is then natural to obtain a complex via the Yoneda interpretation of Ext^2.

We have to introduce some notations

$$G = GL_3(\mathbb{Q}_p) \quad B = \begin{pmatrix} \times & 0 & 0 \\ \times & \times & 0 \\ \times & \times & \times \end{pmatrix} \quad P_1 = \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ \times & \times & \times \end{pmatrix} \quad P_2 = \begin{pmatrix} \times & 0 & 0 \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}$$

$$T = \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix}$$

$\varepsilon_i(\mathrm{diag}(t_0, t_1, t_2)) = t_i$

$\chi_{\vec{w}} = \varepsilon_0^{w_2-2} \varepsilon_1^{w_1-1} : B \twoheadrightarrow T \to \mathbb{Q}_p^\times$

Let $L(\vec{w})$ be the alg. irr. rep of h. wt $(w_2-2, w_1-1, 0)$.

$L(\vec{w}) \hookrightarrow \mathrm{ind}_B^G \chi_{\vec{w}}$ (LA induction)

~~We have~~ Let $L(a,b)$ ———— of GL_2.

$\rho_1 = L(w_2-2, w_1-1) \boxtimes 1$ rep of P_1

$\rho_2 = \varepsilon_0^{w_2-2} \boxtimes L(w_1-1)$ rep of P_2

then $L(\vec{w}) \subset \mathrm{ind}_{P_i}^G \rho_i \subset \mathrm{ind}_B^G \chi_{\vec{w}}$ and we can define

$$\Sigma(\vec{w}) = \frac{\mathrm{ind}_B^G \chi_{\vec{w}}}{\mathrm{ind}_{P_1}^G \rho_1 + \mathrm{ind}_{P_2}^G \rho_2} \quad \text{and} \quad \sigma_i(\vec{w}) = \mathrm{ind}_{P_i}^G(\vec{w}) / L(\vec{w})$$

$$\Sigma(\vec{w}) \supset \mathrm{St} \otimes L(\vec{w}) \qquad \sigma_i(\vec{w}) \supset v_{P_i} \otimes L(\vec{w}).$$

$\uparrow$ smooth gen Steinberg rep.

We have now to compute extensions between all these representations. This computation is possible ~~bec~~ thanks to work of Jan Kohlhaase (he has to consider a bigger exact cat of $D(G,K)$-modules...)

<u>Proposition</u>: • $\mathrm{Ext}^1(\sigma_i(\vec{w}), \Sigma(\vec{w})) \simeq \mathrm{Hom}(\mathbb{Q}_p^\times, K) \simeq K\log_0 \oplus K\mathrm{val}$

$$\begin{array}{ccc} \cup & & \circlearrowleft \\ \mathrm{Ext}^1(v_{P_i}\otimes L(\vec{w}), \mathrm{St}\otimes L(\vec{w})) & \simeq & K\,\mathrm{val} \end{array}$$

• $\mathrm{Ext}^1(L(\vec{w}), \Sigma(\vec{w})) = 0$

• $\mathrm{Ext}^2(L(\vec{w}), \Sigma(\vec{w}))$ is 5-dim

$$\circlearrowleft$$

$$\mathrm{Ext}^2(L(\vec{w}), \mathrm{St}\otimes L(\vec{w})) \simeq K\,\wedge^2\mathrm{val}.$$

Using Yoneda interpretation of Ext^1

$\rightsquigarrow \quad 0 \to \Sigma(\vec{w}) \to \Sigma(\vec{w}, \mathcal{L}, \mathcal{L}') \to \sigma_1(\vec{w}) \oplus \sigma_2(\vec{w}) \to 0$

associated to $(\log_{\mathcal{L}'}, \log_{\mathcal{L}}) = (\log_0 + \mathcal{L}'\mathrm{val}, -)$.

<u>Remark</u>: It is even better to choose

$0 \to \Sigma(\vec{w}, \mathcal{L}, \mathcal{L}') \longrightarrow \tilde{\Sigma}(\vec{w}, \mathcal{L}, \mathcal{L}') \longrightarrow L(\vec{w}) \to 0$

the unique non split extension.

We have now ~~de~~ $\mathrm{Ext}^2(L(\vec{w}), \Sigma(\vec{w}, \mathcal{L}, \mathcal{L}'))$ 2-dim

$$B = \quad \circlearrowleft$$

$$A = \mathrm{Ext}^2(L(\vec{w}), \mathrm{St}\otimes L(\vec{w})).$$

Choose an isomorphisme

$$\Psi: K \xrightarrow{\sim} \mathbb{P}(B) \setminus \mathbb{P}(A).$$

we define $\Sigma(\vec{w}, \underline{\mathcal{L}})$ as being a complex constructing from $\Psi(\mathcal{L}'')$ via Yoneda. It is a object in $\mathcal{D}_G$ the derived cat of Kohlhaase cat of $D(G,K)$-modules.

It is characterized by the property:

we have an ~~exact~~ destinguished triangle such that

$$\Sigma(\vec{w}, \chi, \chi') \to \Sigma(\vec{w}, \vec{\chi}) \to L(\vec{w}')[-1] \to \dots$$

$$\mathrm{Hom}(L(\vec{w}), L(\vec{w}')) \longrightarrow \mathrm{Ext}^2(\cancel{\Sigma(\vec{w}, \chi, \chi')}\ L(\vec{w}'), \Sigma(s\vec{w}, \chi, \chi'))$$

$$\mathrm{id} \longmapsto \Psi(\chi'').$$

<u>How to choose Ψ</u>? (work in progress...)

Now $\vec{w} = (0,1,2)$ $\Sigma(\vec{w})$ is the "analytic" Steinberg St^{an}.

We have an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^2(1, St^{\infty}) \arrow[r] & \mathrm{Ext}^2(1, \Sigma(\chi, \chi')) \arrow[r] & H^3(G,1) \arrow[l, bend right, "?"'] \arrow[dl, bend left, "?"] \arrow[r] & 0 \\
 & & \mathrm{Ext}^2(1, St^{an}) \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

We want to choose a splitting of this exact sequence.

$$0 \to \sigma_2 \to \mathrm{ind}_{P_1}^{G}(St^{an}_{GL_2}) \longrightarrow St^{an} \to 0$$

$$\begin{tikzcd}
\cancel{\mathrm{Ext}^2(1, \ldots)}\ H^2(G, \mathrm{ind}_{P_1}^{G} St^{an}_{GL_2}) \arrow[r] \arrow[d, phantom, "\wr"] & H^3(G,1) \arrow[d, "\mathrm{res}"] \\
H^2(GL_2(\mathbb{Q}_p), St^{an}) \arrow[r] & H^3(GL_2(\mathbb{Q}_p), 1)
\end{tikzcd}$$

$St^{an} =$ "LA functions on $\mathbb{P}^1$ modulo constants".

We can use the Coleman dilogarithm defined by

$$D(z) = li_2(z) + \frac{1}{2}\log(z)\log(1-z).$$

Then consider

$$(g_0, g_1, g_2) \longmapsto \cancel{\mathcal{D}[g} \ (x \longmapsto \mathcal{D}([g_0\infty, g_2\infty, \cancel{g_3\infty}\, x])) \text{ modulo constants.}$$

The cross-product isn't defined everywhere on $(\mathbb{P}^1)^4$, so this is only a measurable cocycle and we have to use the measurable cohomology of C. Moore to define it correctly.

The "good cocycle" should be $\frac{1}{2}\lambda^2 \log + \lambda c_0$, $\lambda \in \mathbb{Q}_p^\times$.

However our last result will not depend on this choice.

$\vec{w} = (0,1,2)$

Let $X = \mathbb{P}^2(\mathbb{C}_p) \setminus \bigcup_{H \ldots} H(\mathbb{C}_p)$ the 2-dim Drinfel'd space. $\circlearrowleft GL_3(\mathbb{Q}_p)$.

It is a rigid $\mathbb{Q}_p$-analytic Stein space.

We can consider the de Rham complex

$$R\Gamma_{dR}(X) = [\mathcal{O}(X) \to \Omega^1(X) \to \Omega^2(X)].$$

It is a complex of coadmissible $D(G,K)$-modules.

What we want now to do is to give a (φ, N)-filtered structure on ~~$\mathrm{Hom}_{D_G}(L(\vec{w}), \Sigma \vec{w}$~~

$$M(\vec{\mathcal{L}}) = \mathrm{Hom}_{D_G}(\Sigma^1(\vec{w}, \vec{\mathcal{L}})'[-1], R\Gamma_{dR}(X)).$$

↳ dual, constructed as $\Sigma(\vec{w}, \vec{\mathcal{L}})$, using

$\mathrm{Ext}^1(A,B) \simeq \mathrm{Ext}^1(B', A')$ for "good" representations (Kohlhaase).

- filtration: we use $\mathrm{Fil}^i R\Gamma_{dR}(X) = [0 \to \dots \to 0 \to \Omega^i \to \dots \to \Omega^2]$.

and $\mathrm{Fil}^i M(\vec{\mathfrak{L}}) = \mathrm{Im}\left(\mathrm{Hom}_{D_0}\left(\Sigma(\vec{\mathfrak{L}})^\vee[-i], \mathrm{Fil}^i R\Gamma_{dR} \right) \to M(\vec{\mathfrak{L}}) \right)$.

- φ and N. Choose a splitting

$$R\Gamma_{dR}(X) \xrightarrow[\sim]{s} \bigoplus H^i_{dR}[-i] \simeq \underbrace{H^0_{dR} \oplus H^1_{dR}[-1]}_{\xleftarrow{N}} \oplus H^2_{dR}[-2]$$

and define $\varphi = \oplus\, p^i[-i]$

and $N \in \mathrm{Ext}^1(H^2_{dR}(X), H^1_{dR}) \times \mathrm{Ext}^1(H^1_{dR}, H^0_{dR})$

of max rank. $\to M(\vec{\mathfrak{L}})_s$ (φ, N)-filtered module.

<u>Theorem</u>: $\exists !\, s$ such that $\forall \vec{\mathfrak{L}} \in K^3$,

$$M(\vec{\mathfrak{L}})_s \simeq D(\vec{w}, \vec{\mathfrak{L}}).$$